Mathematicians find new solutions to an ancient puzzle

Many people find complex math puzzling, including some mathematicians. Recently, mathematician Daniel J. Madden and retired physicist, Lee W. Jacobi, found solutions to a puzzle that has been around for centuries.

Jacobi and Madden have found a way to generate an infinite number of solutions for a puzzle known as 'Euler's Equation of degree four.'

The equation is part of a branch of mathematics called number theory. Number theory deals with the properties of numbers and the way they relate to each other. It is filled with problems that can be likened to numerical puzzles.

"It's like a puzzle: can you find four fourth powers that add up to another fourth power" Trying to answer that question is difficult because it is highly unlikely that someone would sit down and accidentally stumble upon something like that," said Madden, an associate professor of mathematics at The University of Arizona in Tucson.

The team's finding is published in the March issue of *The American Mathematical Monthly*.

Equations are puzzles that need certain solutions "plugged into them" in order to create a statement that obeys the rules of logic.

For example, think of the equation x + 2 = 4. Plugging "3" into the equation doesn't work, but if x = 2, then the equation is correct.

In the mathematical puzzle that Jacobi and Madden worked on, the problem was finding variables that satisfy a Diophantine equation of order four. These equations are so named because they were first studied by the ancient Greek mathematician Diophantus, known as 'the father of algebra.'

In its most simple version, the puzzle they were trying to solve is the equation:
(a)(to the fourth power) + (b)(to the fourth power) + (c)(to the fourth power) + (d)(to the fourth power) = (a + b + c + d)(to the fourth power)

That equation, expressed mathematically, is:
a4 + b4 +c4 +d4 = (a + b + c + d)4

Madden and Jacobi found a way to find the numbers to substitute, or plug in, for the a's, b's, c's and d's in the equation. All the solutions they have found so far are very large numbers.

In 1772, Euler, one of the greatest mathematicians of all time, hypothesized that to satisfy equations with higher powers, there would need to be as many variables as that power. For example, a fourth order equation would need four different variables, like the equation above.

Euler's hypothesis was disproved in 1987 by a Harvard graduate student named Noam Elkies. He found a case where only three variables were needed. Elkies solved the equation: (a)(to the fourth power) + (b)(to the fourth power) + (c)(to the fourth power) = e(to the fourth power), which shows only three variables are needed to create a variable that is a fourth power.

Inspired by the accomplishments of the 22-year-old graduate student, Jacobi began working on mathematics as a hobby after he retired from the defense industry in 1989.

Fortunately, this was not the first time he had dealt with Diophantine equations. He was familiar with them because they are commonly used in physics for calculations relating to string theory.

Jacobi started searching for new solutions to the puzzle using methods he found in some number theory texts and academic papers.

He used those resources and Mathematica, a computer program used for mathematical manipulations.

Jacobi initially found a solution for which each of the variables was 200 digits long. This solution was different from the other 88 previously known solutions to this puzzle, so he knew he had found something important.

Jacobi then showed the results to Madden. But Jacobi initially miscopied a variable from his Mathematica computer program, and so the results he showed Madden were incorrect.

“The solution was wrong, but in an interesting way. It was close enough to make me want to see where the error occurred,” Madden said.

When they discovered that the solution was invalid only because of Jacobi’s transcription error, they began collaborating to find more solutions.

Madden and Jacobi used elliptic curves to generate new solutions. Each solution contains a seed for creating more solutions, which is much more efficient than previous methods used.

In the past, people found new solutions by using computers to analyze huge amounts of data. That required a lot of computing time and power as the magnitude of the numbers soared.

Now people can generate as many solutions as they wish. There are an infinite number of solutions to this problem, and Madden and Jacobi have found a way to find them all.

The title of their paper is, “On a4 + b4 +c4 +d4 = (a + b + c + d)4."

“Modern number theory allowed me to see with more clarity the implications of his (Jacobi’s) calculations,” Madden said.

“It was a nice collaboration,” Jacobi said. “I have learned a certain amount of new things about number theory; how to think in terms of number theory, although sometimes I can be stubbornly algebraic.”

Source: University of Arizona