Mathematicians unlock major number theory puzzle

Mathematicians have finally laid to rest the legendary mystery surrounding an elusive group of numerical expressions known as the "mock theta functions." Number theorists have struggled to understand the functions ever since the great Indian mathematician Srinivasa Ramanujan first alluded to them in a letter written on his deathbed, in 1920.

Now, using mathematical techniques that emerged well after Ramanujan's death, two number theorists at the University of Wisconsin-Madison have pieced together an explanatory framework that for the first time illustrates what mock theta functions are, and exactly how to derive them.

Their new theory is proving invaluable in the resolution of long-standing open questions in number theory. In addition, the UW-Madison advance will for the first time enable researchers to apply mock theta functions to problems in a variety of fields, including physics, chemistry and several branches of mathematics. The findings appear in a series of three papers, the third appearing today in the *Proceedings of the National Academy of Sciences*.

"It's extremely gratifying to be able to say we solved the 'final problem' of Ramanujan," says co-author Ken Ono, UW-Madison Manasse Professor of Letters and Science, who is widely noted for contributions to number theory. "We simply got really lucky."

Ono worked in collaboration with German mathematician Kathrin Bringmann, a postdoctoral researcher at UW-Madison.

"This is something I really didn't expect anybody to do," says George Andrews, a leading number theorist at Pennsylvania State University who in 2000 called mock theta functions one of the most difficult math puzzles of the new millennium. "It is an outstanding piece of work, a breathtakingly wonderful achievement."

Working from Ramanujan's letter, number theorists believed that mock theta functions are related to a well-understood class of mathematical expressions-the 'theta' functions-that have been in use for centuries. Theta functions constitute a certain sequence of numbers that has proved useful in various problems of mathematical analysis.

Mock theta functions similarly constitute an infinite series of numbers. But what has been completely baffling is what it is about mock theta series that make them so rich and powerful. Over the decades-much to the amazement of mathematicians everywhere-mock theta functions have cropped up amidst calculations in a number of fields, including mathematics, physics, chemistry, and even cancer research.

What made mock theta functions all the more inscrutable was the fact that the first few pages of Ramanujan's letter were lost. Those pages may have contained more clues, but in their absence, the letter merely presented 17 examples of the functions. What's missing is any definition of what the functions are, any hints on how to derive them, and any indication of why they are even important. All those secrets died with Ramanujan just two months after he wrote the letter, when he succumbed to tuberculosis at the age of 32.

"Imagine stringing together a thousand random words and then saying you've come up with the most beautiful poetry," says Ono. "That's essentially what Ramanujan did to us."

Bringmann and Ono made sense of it all by finding a way to represent the power of mock theta functions through another relatively new family of mathematical expressions known as the Harmonic Maass Forms.

A Dutch mathematician named Sander Zwegers had already made that important connection in 2002, but he had focused only on Ramanujan's examples.

It was during a flight to New Hampshire that Ono realized the full depth and meaning of Zwegers' work. Skimming a journal to pass the time, Ono happened upon an old article by George Andrews on mock theta functions. Suddenly, he noticed that some of the mathematics in the paper seemed to resonate with parts of the Harmonic Maass theory, which he and Bringmann just happened to be developing at the time, for other reasons.

The mathematicians found the connection held up beautifully. "We knew we were onto something right away," says Ono. "It was an uncanny set of coincidences that lead us to this solution. It was as if it all just fell into our lap and now we are serendipitously applying our theory to longstanding open problems."

Source: University of Wisconsin-Madison